

# THE BISHOP-PHELPS-BOLLOBÁS THEOREM FOR OPERATORS FROM $\ell_1$ SUMS OF BANACH SPACES

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ABSTRACT. We introduce a generalized approximate hyperplane series property for a pair  $(X, Y)$  of Banach spaces to characterize when  $(\ell_1(X), Y)$  has the Bishop-Phelps-Bollobás property. In particular, we show that  $(X, Y)$  has this property if  $X, Y$  are finite-dimensional, if  $X$  is a  $C(K)$  space and  $Y$  is a Hilbert space, or if  $X$  is Asplund and  $Y = C_0(L)$ , where  $K$  is a compact Hausdorff space and  $L$  is a locally compact Hausdorff space.

## 1. INTRODUCTION

Let  $X$  be a real or complex Banach space.  $B_X$ ,  $S_X$  and  $X^*$  denote the closed unit ball, the unit sphere and the topological dual of  $X$ , respectively. By  $\mathcal{L}(X, Y)$  we denote the space of all bounded linear operators between Banach spaces  $X$  and  $Y$ .

The starting point of the study of density of norm attaining operators is the famous Bishop-Phelps theorem [7] of 1961, which states that the set of norm attaining functionals on a Banach space is dense in its dual space. Afterwards, there have been a lot of efforts to extend this theorem to operators. Especially, this study is deeply influenced by the work [18] of Lindenstrauss. One of the milestones of this theory is the result of Bourgain [9]. He showed that every bounded linear operator whose domain is a Banach space with the Radon-Nikodým property can be approximated by norm-attaining operators and, conversely, if this property holds in every equivalent norm then the space has the Radon-Nikodým property. We refer to the survey paper [1] for a detailed account on the theory of norm attaining operators. On the other hand, motivated by the study of numerical ranges of operators, Bollobás [8] proved in 1970 a refinement of the Bishop-Phelps theorem, nowadays known as the Bishop-Phelps-Bollobás theorem [8, Theorem 1]:

Let  $X$  be a Banach space. Suppose  $x \in S_X$  and  $x^* \in S_{X^*}$  satisfy  $|x^*(x) - 1| < \varepsilon^2/2$  for some  $0 < \varepsilon < 1/2$ . Then there exist  $y \in S_X$  and  $y^* \in S_{X^*}$  satisfying that  $y^*(y) = 1$ ,  $\|y - x\| < \varepsilon + \varepsilon^2$  and  $\|y^* - x^*\| < \varepsilon$ .

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Carrying Bollobás' ideas to the vector-valued case, Acosta, Aron, García and Maestre [3] introduced in 2008 the Bishop-Phelps-Bollobás property for operators, as follows.

**Definition 1** ([3]). Let  $X$  and  $Y$  be (real or complex) Banach spaces. We say that the pair  $(X, Y)$  has the *Bishop-Phelps-Bollobás property* (in short, *BPBp*) if, given  $0 < \varepsilon < 1$ , there is  $\eta(\varepsilon) > 0$  such that whenever  $T \in S_{\mathcal{L}(X, Y)}$  and  $x \in S_X$  satisfy  $\|Tx\| > 1 - \eta(\varepsilon)$ , then there exist  $z \in S_X$  and  $S \in S_{\mathcal{L}(X, Y)}$  such that

$$\|Sz\| = 1, \quad \|x - z\| < \varepsilon \quad \text{and} \quad \|T - S\| < \varepsilon.$$

In this case, we will say that  $(X, Y)$  has the BPBp with the function  $\varepsilon \mapsto \eta(\varepsilon)$ .

This property has attracted the attention of many researchers. The list of papers dealing with the BPBp contains [2, 4, 5, 6, 10, 11, 15, 17], where we refer for more information on recent results.

One of the most surprising result concerning the BPBp presented in the seminal paper [3] is the existence of a Banach space  $Y$  such that the pair  $(\ell_1, Y)$  fails the BPBp, even though the set of norm attaining operators from  $\ell_1$  to any Banach space  $Y$  is dense in  $\mathcal{L}(\ell_1, Y)$  (see [18, 9]). Let us also comment that, even more surprising, there are pairs  $(X, Y)$  of Banach spaces failing the BPBp in which  $X$  is finite-dimensional [3], even when every linear operator whose domain is a finite-dimensional space attains its norm. For instance, it is known [6, Corollary 3.5] that if  $X$  is a two-dimensional real Banach space which is not uniformly convex, then there is a Banach space  $Y$  such that the pair  $(X, Y)$  fails the BPBp. In the cited paper [3], a property called approximate hyperplane series property was introduced to characterize those Banach spaces  $Y$  such that  $(\ell_1, Y)$  has the BPBp.

**Definition 2** ([3]). A Banach space  $Y$  has the *approximate hyperplane series property* (*AHSP*, in short) if for every  $\varepsilon > 0$  there exists  $\eta > 0$  such that given a sequence  $(y_k) \subset S_Y$  and a convex series  $\sum_{k=1}^{\infty} \alpha_k$  such that

$$\left\| \sum_{k=1}^{\infty} \alpha_k y_k \right\| > 1 - \eta,$$

there exist  $A \subset \mathbb{N}$ ,  $y^* \in S_{Y^*}$ , and a subset  $\{z_k : k \in A\} \subset S_Y$  satisfying that

$$\sum_{k \in A} \alpha_k > 1 - \varepsilon, \quad \|z_k - y_k\| < \varepsilon \quad \text{and} \quad y^*(z_k) = 1$$

for every  $k \in A$ .

Among the spaces with the AHSP, we may cite finite dimensional spaces, uniformly convex spaces,  $C_0(L)$  spaces and  $L_1(\mu)$  spaces, as representative examples [3]. On the other hand, there are spaces failing this property: every strictly convex space which is not uniformly convex [3] and a particular polyhedral space constructed in [6, §4]. We refer the reader to the paper [12, 13] for more examples of spaces with the AHSP.

In this paper, we first introduce a property for a pair  $(X, Y)$  of Banach spaces called the generalized approximate hyperplane series property (abbreviated to *generalized AHSP*, see Definition 4) to characterize when  $(\ell_1(X), Y)$  has the BPBp. This characterization is used to present a number of pairs  $(X, Y)$  of Banach spaces such that  $(\ell_1(X), Y)$  has the BPBp. Namely, (1) when  $X$  and  $Y$  are finite dimensional, (2) when  $Y$  is a Hilbert space and  $(X, Y)$  has the BPBp, in particular if  $X = C(K)$  or  $X$  is uniformly convex, and (3) if  $X$  is Asplund and  $Y$  is a uniform algebra, in particular,  $Y = C_0(L)$ .

We finish this introduction with an easy result from [3] about convex series for later use.

**Lemma 3** ([3, Lemma 3.3]). *Let  $\{c_n\}$  be a sequence of scalars with  $|c_n| \leq 1$  for every  $n$ , and let  $\sum_{n=1}^{\infty} \alpha_n$  be a convex series such that  $\operatorname{Re} \sum_{n=1}^{\infty} \alpha_n c_n > 1 - \eta$  for some  $\eta > 0$ . Then for every  $0 < r < 1$ , the set  $A = \{i \in \mathbb{N} : \operatorname{Re} c_i > r\}$  satisfies the estimate  $\sum_{i \in A} \alpha_i \geq 1 - \frac{\eta}{1-r}$ .*

## 2. THE RESULTS

We begin with the definition of the *generalized AHSP*.

**Definition 4.** A pair of Banach spaces  $(X, Y)$  is said to have the *generalized AHSP* if for every  $\varepsilon > 0$  there exists  $0 < \eta(\varepsilon) < \varepsilon$  such that given sequences  $(T_k) \subset S_{\mathcal{L}(X, Y)}$  and  $(x_k) \subset S_X$  and a convex series  $\sum_{k=1}^{\infty} \alpha_k$  such that

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \eta(\varepsilon),$$

there exist a subset  $A \subset \mathbb{N}$ ,  $y^* \in S_{Y^*}$  and sequences  $(S_k) \subset S_{\mathcal{L}(X, Y)}$ ,  $(z_k) \subset S_X$  satisfying the following:

- (1)  $\sum_{k \in A} \alpha_k > 1 - \varepsilon$ ,
- (2)  $\|z_k - x_k\| < \varepsilon$  and  $\|S_k - T_k\| < \varepsilon$  for all  $k \in A$ ,
- (3)  $y^*(S_k z_k) = 1$  for every  $k \in A$ .

Some observations are pertinent.

**Remark 5.** Let  $X, Y$  be Banach spaces.

- (a) *To show that the pair  $(X, Y)$  has the generalized AHSP it is enough to check the conditions for every finite (but of arbitrarily length) convex series, with the same function  $\varepsilon \mapsto \eta(\varepsilon)$  for all lengths.*
- (b) *In the definition of the generalized AHSP we may consider sequences  $(T_k) \subset B_{\mathcal{L}(X, Y)}$  and  $(x_k) \subset B_X$  (with a small change in the function  $\eta(\varepsilon)$ ).*
- (c) *Since we may consider  $\alpha_1 = 1$ , we obtain that if  $(X, Y)$  has the generalized AHSP, then  $(X, Y)$  has the BPBp.*
- (d) *Also, if  $(X, Y)$  has the generalized AHSP, then  $Y$  has the AHSP. This follows easily by replacing the operators  $T_k$ 's with suitable rank-one operators.*

*Proof.* Only part (b) needs a detailed proof. Suppose that  $X$  has the generalized AHSP with a function  $\varepsilon \mapsto \eta(\varepsilon)$ , and write

$$\eta'(\varepsilon) = \frac{\eta\left(\frac{\varepsilon}{2}\right) \varepsilon^2}{8}$$

for every  $\varepsilon \in (0, 1)$ . Fix  $\varepsilon \in (0, 1)$  and consider sequences  $(T_n)$  in  $B_{\mathcal{L}(X, Y)}$  and  $(x_n)$  in  $B_X$  and a convex series  $\sum_n \alpha_n$  such that

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \eta'(\varepsilon).$$

Pick  $y_0^* \in S_{Y^*}$  such that

$$\operatorname{Re} y_0^* \left( \sum_{k=1}^{\infty} \alpha_k T_k x_k \right) = \sum_{k=1}^{\infty} \alpha_k \operatorname{Re} y_0^*(T_k x_k) > 1 - \eta'(\varepsilon)$$

and define

$$M := \left\{ k \in \mathbb{N} : \operatorname{Re} y_0^*(T_k x_k) > 1 - \frac{\varepsilon}{2} \right\}.$$

By Lemma 3, we have that

$$\sum_{k \in M^c} \alpha_k < \frac{2\eta'(\varepsilon)}{\varepsilon} = \frac{\eta\left(\frac{\varepsilon}{2}\right)\varepsilon}{4}.$$

Now, for every  $k \in \mathbb{N}$  we consider  $T'_k \in S_{\mathcal{L}(X, Y)}$  and  $x'_k \in S_X$  such that

$$T'_k \|T_k\| = T_k \quad \text{and} \quad x'_k \|x_k\| = x_k.$$

We observe that

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \alpha_k T'_k x'_k \right\| &\geq \sum_{k=1}^{\infty} \alpha_k \operatorname{Re} y_0^*(T'_k x'_k) \\ &\geq \sum_{k \in M} \alpha_k \operatorname{Re} y_0^* \left( \frac{T_k}{\|T_k\|} \frac{x_k}{\|x_k\|} \right) - \sum_{k \in M^c} \alpha_k \\ &\geq \sum_{k \in M} \alpha_k \operatorname{Re} y_0^*(T_k x_k) - \sum_{k \in M^c} \alpha_k \\ &\geq \sum_{k=1}^{\infty} \alpha_k \operatorname{Re} y_0^*(T_k x_k) - 2 \sum_{k \in M^c} \alpha_k \\ &> 1 - \eta'(\varepsilon) - \frac{\eta\left(\frac{\varepsilon}{2}\right)\varepsilon}{2} = 1 - \frac{\eta\left(\frac{\varepsilon}{2}\right)\varepsilon^2}{8} - \frac{\eta\left(\frac{\varepsilon}{2}\right)\varepsilon}{2} > 1 - \eta\left(\frac{\varepsilon}{2}\right). \end{aligned}$$

By the generalized AHSP of the pair  $(X, Y)$ , there exist  $A' \subset \mathbb{N}$ ,  $y^* \in S_{Y^*}$  and sequences  $(S_k) \subset S_{\mathcal{L}(X, Y)}$ ,  $(z_k) \subset S_X$  satisfying  $\sum_{k \in A'} \alpha_k > 1 - \frac{\varepsilon}{2}$  and

$$y^*(S_k z_k) = 1, \quad \|z_k - x'_k\| < \frac{\varepsilon}{2}, \quad \|S_k - T'_k\| < \frac{\varepsilon}{2} \quad (k \in A').$$

Now, set  $A := A' \cap M$  and observe that

$$\sum_{k \in A} \alpha_k \geq \sum_{k \in A'} \alpha_k - \sum_{k \in M^c} \alpha_k > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon.$$

On the other hand, for every  $k \in A$ , we have  $\|T_k\| > 1 - \varepsilon/2$  and  $\|z_k\| > 1 - \varepsilon/2$ , implying that  $T'_k = \frac{T_k}{\|T_k\|}$  and  $x'_k = \frac{x_k}{\|x_k\|}$  and, moreover, that

$$\|S_k - T_k\| < \varepsilon \quad \text{and} \quad \|z_k - x_k\| < \varepsilon,$$

finishing the proof.  $\square$

Our first aim in this section is to show that the *generalized AHSP* characterizes a pair  $(X, Y)$  of Banach spaces for which  $(\ell_1(X), Y)$  has the BPBp.

**Theorem 6.** *Let  $X$  and  $Y$  be Banach spaces. Then the following are equivalent.*

- (i) *The pair  $(X, Y)$  has the generalized AHSP,*
- (ii) *the pair  $(\ell_1(X), Y)$  has the BPBp,*
- (iii) *there is a function  $\eta : (0, 1) \rightarrow (0, \infty)$  such that  $(\ell_1^n(X), Y)$  has the BPBp with the function  $\eta$  for every  $n \in \mathbb{N}$ .*

*Proof.* (i)  $\Rightarrow$  (ii). Fix  $0 < \varepsilon < 1$ . Let  $\eta(\varepsilon)$  be given by the definition of the generalized AHSP, and set  $\rho(\varepsilon) := \eta(\varepsilon/3)$ . Suppose that a bounded linear operator  $T : \ell_1(X) \rightarrow Y$  and  $x \in S_{\ell_1(X)}$  satisfy that

$$\|T\| = 1 \quad \text{and} \quad \|Tx\| > 1 - \rho(\varepsilon).$$

Let  $T_k$  be the restriction of  $T$  on the  $k$ -th coordinate of  $\ell_1(X)$  and observe that  $\|T\| = \sup_{k \in \mathbb{N}} \|T_k\|$ . Write  $x = (\alpha_k x_k)$  for some  $x_k \in S_X$  and a convex series  $\sum_{k=1}^{\infty} \alpha_k$ . Since  $Tx = \sum_{k=1}^{\infty} \alpha_k T_k x_k$ , we have

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \rho(\varepsilon) \geq 1 - \eta(\varepsilon/3).$$

From our assumption (using Remark 5.b), there exist a subset  $A \subset \mathbb{N}$ , a functional  $y^* \in S_{Y^*}$  and sequences  $(S_k) \subset S_{\mathcal{L}(X,Y)}$ ,  $(z_k) \subset S_X$  satisfying for all  $k \in A$  that

$$\sum_{k \in A} \alpha_k > 1 - \frac{\varepsilon}{3}, \quad \|z_k - x_k\| < \frac{\varepsilon}{3}, \quad \|T_k - S_k\| < \frac{\varepsilon}{3} \quad \text{and} \quad y^*(S_k z_k) = 1.$$

Define  $S : \ell_1(X) \rightarrow Y$  by

$$S(y) = \sum_{k \in A} S_k y_k + \sum_{k \in \mathbb{N} \setminus A} T_k y_k \quad \text{for every } y = (y_k) \in \ell_1(X)$$

and observe that  $\|S\| = 1$  and

$$\|S - T\| = \sup_{k \in A} \|S_k - T_k\| < \frac{\varepsilon}{3} < \varepsilon.$$

Write  $z_k = 0$  for each  $k \in \mathbb{N} \setminus A$  and let  $z = (\beta_k z_k)$ , where

$$\beta_k = \frac{\alpha_k}{\sum_{k \in A} \alpha_k} \quad \text{for } k \in A \quad \text{and} \quad \beta_k = 0 \quad \text{for } k \in \mathbb{N} \setminus A.$$

Next, we have the estimate

$$\begin{aligned} \|z - x\| &\leq \left\| z - \left( \sum_{k \in A} \alpha_k \right) z \right\| + \left\| \left( \sum_{k \in A} \alpha_k \right) z - x \right\| \\ &\leq \sum_{k \in \mathbb{N} \setminus A} \alpha_k + \sum_{k \in A} \alpha_k \|z_k - x_k\| + \sum_{k \in \mathbb{N} \setminus A} \alpha_k < \varepsilon. \end{aligned}$$

Finally, we get that

$$1 \geq \|Sz\| \geq |y^*(Sz)| \geq \left| \sum_{k \in A} \beta_k y^*(S_k z_k) \right| = 1,$$

so  $\|Sz\| = 1$  and we are done.

(ii)  $\Rightarrow$  (iii). This follows from [6, Theorem 2.1], taking into account that for every  $n \in \mathbb{N}$ ,  $\ell_1^n(X) \equiv \ell_1^n(X) \oplus_1 \ell_1(X)$ .

(iii)  $\Rightarrow$  (i). Fix  $\varepsilon > 0$ . We choose  $\delta(\varepsilon) > 0$  small enough to be  $\frac{2\delta(\varepsilon)}{\varepsilon} + \delta(\varepsilon) < \varepsilon$ .

By Remark 5.a, we may prove that the pair  $(X, Y)$  has the generalized AHSP by just considering finite convex series. Consider finite sequences  $(T_k)_{k=1}^n \subset S_{\mathcal{L}(X,Y)}$ ,  $(x_k)_{k=1}^n \subset S_X$  and a finite convex series  $\sum_{k=1}^n \alpha_k$  satisfying

$$\left\| \sum_{k=1}^n \alpha_k T_k x_k \right\| > 1 - \eta(\delta(\varepsilon)).$$

We define a bounded linear operator  $T : \ell_1^n(X) \rightarrow Y$  by

$$T(y) = \sum_{k=1}^n T_k z_k \quad \text{for every } z = (z_k)_{k=1}^n \in \ell_1^n(X).$$

We write  $N = \{1, \dots, n\}$  and  $x = (\alpha_k x_k)_{k=1}^n \in \ell_1^n(X)$ . We clearly have that  $\|T\| = 1$  and  $\|Tx\| > 1 - \eta(\delta(\varepsilon))$ . By (iii), there exist a bounded linear operator  $S : \ell_1^n(X) \rightarrow Y$  and  $z = (z_k) \in S_{\ell_1^n(X)}$  such that

$$\|S\| = \|Sz\| = 1, \quad \|T - S\| < \delta(\varepsilon), \quad \text{and} \quad \|x - z\| < \delta(\varepsilon).$$

Write  $B = \{k \in N : \|z_k\| = 0\}$  and observe that

$$\delta(\varepsilon) > \|x - z\| = \sum_{k=1}^n \|\alpha_k x_k - z_k\| \geq \sum_{k \in B} \alpha_k.$$

We also have that

$$\begin{aligned} \|x - z\| &\geq \sum_{k \in N \setminus B} \|\alpha_k x_k - z_k\| = \sum_{k \in N \setminus B} \alpha_k \left\| x_k - \frac{\|z_k\|}{\alpha_k} \frac{z_k}{\|z_k\|} \right\| \\ &\geq \sum_{k \in N \setminus B} \alpha_k \left( \left\| x_k - \frac{z_k}{\|z_k\|} \right\| - \left\| \frac{z_k}{\|z_k\|} - \frac{\|z_k\|}{\alpha_k} \frac{z_k}{\|z_k\|} \right\| \right), \end{aligned}$$

and for every  $k \in N \setminus B$  we have

$$\begin{aligned} \left\| \frac{z_k}{\|z_k\|} - \frac{\|z_k\|}{\alpha_k} \frac{z_k}{\|z_k\|} \right\| &= \left| 1 - \frac{\|z_k\|}{\alpha_k} \right| = \left| \|x_k\| - \frac{\|z_k\|}{\alpha_k} \right\| \left\| \frac{z_k}{\|z_k\|} \right\| \\ &\leq \left\| x_k - \frac{\|z_k\|}{\alpha_k} \frac{z_k}{\|z_k\|} \right\|. \end{aligned}$$

Hence, we get that

$$\sum_{k \in N \setminus B} \alpha_k \left\| x_k - \frac{z_k}{\|z_k\|} \right\| \leq 2\|x - z\| < 2\delta(\varepsilon).$$

Set  $A = \{k \in N : \|z_k\| \neq 0, \|x_k - \frac{z_k}{\|z_k\|}\| < \varepsilon\}$ . Then we obtain that

$$2\delta(\varepsilon) > \sum_{k \in N \setminus B} \alpha_k \left\| x_k - \frac{z_k}{\|z_k\|} \right\| \geq \sum_{k \in (N \setminus B) \setminus A} \alpha_k \left\| x_k - \frac{z_k}{\|z_k\|} \right\| \geq \varepsilon \sum_{k \in (N \setminus B) \setminus A} \alpha_k,$$

which implies

$$\sum_{k \in A} \alpha_k = 1 - \sum_{k \in B} \alpha_k - \sum_{k \in (N \setminus B) \setminus A} \alpha_k > 1 - \delta(\varepsilon) - \frac{2\delta(\varepsilon)}{\varepsilon} > 1 - \varepsilon.$$

Now, by the Hahn Banach Theorem, there exists  $y^* \in S_{Y^*}$  such that

$$1 = \|S\| = \|Sz\| = y^*(Sz) = y^* \left( \sum_{k=1}^n S_k z_k \right) = \sum_{k=1}^n y^*(S_k z_k),$$

where  $S_k$  is the restriction of  $S$  on  $k$ -th coordinate of  $\ell_1^n(X)$  for every  $k \in N$ . Since  $1 = \|z\| = \sum_{k=1}^n \|z_k\|$ , we get  $y^*(S_k z_k) = \|z_k\|$  for every  $k \in N$ , which means  $y^* \left( S_k \frac{z_k}{\|z_k\|} \right) = 1$  for every  $k \in N$  such that  $\|z_k\| \neq 0$ , in particular, for every  $k \in A$ . Finally, it also follows that  $\|S_k\| = 1$  and  $\|T_k - S_k\| \leq \|T - S\| < \delta(\varepsilon) < \varepsilon$  for each  $k \in A$ . By definition of the set  $A$ , we also have  $\left\| x_k - \frac{z_k}{\|z_k\|} \right\| < \varepsilon$  for every  $k \in A$ , finishing the proof.  $\square$

In the following, we use the above characterization to provide examples of pairs  $(X, Y)$  such that  $(\ell_1(X), Y)$  have the Bishop-Phelps-Bollobás property.

We start with the simple case in which  $X$  and  $Y$  are finite-dimensional.

**Proposition 7.** *For every finite dimensional spaces  $X$  and  $Y$ , the pair  $(X, Y)$  has the generalized AHSP. Equivalently,  $(\ell_1(X), Y)$  has the BPBp.*

We begin with a preliminary lemma.

**Lemma 8.** *Let  $X$  and  $Y$  be any finite dimensional spaces. Then for each  $\varepsilon > 0$ , there is  $\eta > 0$  satisfying the following: given  $y_0^* \in S_{Y^*}$ , there exists  $y_1^* \in S_{Y^*}$  such that whenever  $x_0 \in S_X$  and  $T_0 \in S_{\mathcal{L}(X,Y)}$  satisfy  $\operatorname{Re} y_0^*(T_0 x_0) > 1 - \eta$ , there exist  $x_1 \in S_X$  and  $T_1 \in S_{\mathcal{L}(X,Y)}$  such that*

$$y_1^*(T_1 x_1) = 1, \quad \|T_0 - T_1\| < \varepsilon \quad \text{and} \quad \|x_0 - x_1\| < \varepsilon.$$

*Proof.* We argue by contradiction. Assume there is  $\varepsilon > 0$  which does not satisfy the conditions. Then, for every  $n \in \mathbb{N}$  there is  $z_n^* \in S_{Y^*}$  satisfying that for each  $y^* \in S_{Y^*}$  there exists  $(z_0, S_0) \in S_X \times S_{\mathcal{L}(X,Y)}$  such that  $\operatorname{Re} z_n^*(S_0 z_0) > 1 - \frac{1}{n}$  and  $\max(\|S_0 - S_1\|, \|z_0 - z_1\|) \geq \varepsilon$  whenever  $(z_1, S_1) \in S_X \times S_{\mathcal{L}(X,Y)}$  and  $y^*(S_1 z_1) = 1$ .

By finite dimensionality, we may assume that  $(z_n^*)$  converges to  $z^* \in S_{Y^*}$ . From the above observation, for each  $n$  we may find  $(x_n, T_n) \in S_X \times S_{\mathcal{L}(X,Y)}$  such that

$$\operatorname{Re} z_n^*(T_n x_n) > 1 - \frac{1}{n} \quad \text{and} \quad \max(\|T_n - S\|, \|x_n - z\|) \geq \varepsilon$$

for every  $(z, S) \in S_X \times S_{\mathcal{L}(X,Y)}$  with  $z^*(Sz) = 1$ . Since  $X, Y$  are finite dimensional, we may assume that  $(x_n)$  and  $(T_n)$  converge to  $x \in S_X$  and  $T \in S_{\mathcal{L}(X,Y)}$ , respectively, and so we get that  $z^*Tx = 1$ . However we have  $\max(\|T_n - T\|, \|x_n - x\|) \geq \varepsilon$  for all  $n$ , which contradicts the fact that  $x_n$  and  $T_n$  converge to  $x$  and  $T$ , respectively.  $\square$

*Proof of Proposition 7.* For fixed  $0 < \varepsilon < 1$ , choose  $0 < \eta < \varepsilon$  which satisfies the condition in Lemma 8. Suppose that the sequences  $(T_k) \subset S_{\mathcal{L}(X,Y)}$ ,  $(x_k) \subset S_X$  and the convex series  $\sum_{k=1}^{\infty} \alpha_k$  satisfy

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \eta^2.$$

Choose  $z^* \in S_{Y^*}$  such that

$$\operatorname{Re} z^* \left( \sum_{k=1}^{\infty} \alpha_k T_k x_k \right) = \sum_{k=1}^{\infty} \alpha_k \operatorname{Re} z^*(T_k x_k) > 1 - \eta^2,$$

and set  $A := \{k \in \mathbb{N} : \operatorname{Re} z^*(T_k x_k) > 1 - \eta\}$ . Then we get from Lemma 3 that

$$\sum_{k \in A} \alpha_k > 1 - \eta > 1 - \varepsilon.$$

Now, Lemma 8 gives the existence of  $y^* \in S_{Y^*}$  and  $(z_k, S_k) \in S_X \times S_{\mathcal{L}(X,Y)}$  for every  $k \in A$  satisfying that

$$y^*(S_k z_k) = 1 \quad \text{and} \quad \max(\|T_k - S_k\|, \|x_k - z_k\|) < \varepsilon \quad \text{for every } k \in A. \quad \square$$

When the range space is a Hilbert space, then we have the following.

**Proposition 9.** *Let  $H$  be a Hilbert space and let  $X$  be a Banach space. If  $(X, H)$  has the BPBp, then it has the generalized AHSP, so  $(\ell_1(X), H)$  has the BPBp.*

*Proof.* Let  $\varepsilon \mapsto \delta(\varepsilon)$  and  $\varepsilon \mapsto \eta(\varepsilon)$  be, respectively, the modulus of convexity of  $H$  and the function given by the fact that  $(X, H)$  has the BPBp. Fix  $0 < \varepsilon < 1$  and choose  $\varepsilon' > 0$  such that

$$\varepsilon' < \frac{1}{3} \delta\left(\frac{\varepsilon}{2}\right) \quad \text{and} \quad \eta(\varepsilon') < \frac{1}{3} \delta\left(\frac{\varepsilon}{2}\right).$$

Suppose that the sequences  $(T_k) \subset S_{\mathcal{L}(X,H)}$ ,  $(x_k) \subset S_X$  and the convex series  $\sum_{k \in \mathbb{N}} \alpha_k$  satisfy that

$$\left\| \sum_{k=1}^{\infty} \alpha_k T_k x_k \right\| > 1 - \eta(\varepsilon')^2.$$

Choose  $y^* \in S_{H^*}$  and  $y \in H$  such that

$$\operatorname{Re} y^* \left( \sum \alpha_k T_k x_k \right) > 1 - \eta(\varepsilon') \quad \text{and} \quad y^*(y) = 1.$$

Define the set  $A := \{k \in \mathbb{N} : \operatorname{Re} y^* T_k x_k > 1 - \eta(\varepsilon')\}$  and observe that

$$\sum_{k \in A} \alpha_k > 1 - \eta(\varepsilon')$$

by Lemma 3. Since  $(X, H)$  has the BPBp, for each  $k \in A$  there exist  $S'_k \in S_{\mathcal{L}(X,H)}$  and  $z_k \in S_X$  such that

$$\|S'_k - T_k\| < \varepsilon', \quad \|z_k - x_k\| < \varepsilon' \quad \text{and} \quad \|S'_k z_k\| = 1$$

for every  $k \in A$ . Now, for every  $k \in A$  we have

$$\begin{aligned} \operatorname{Re} y^* S'_k z_k &\geq \operatorname{Re} y^* T_k x_k - \|T_k - S'_k\| - \|x_k - z_k\| \\ &> 1 - \eta(\varepsilon') - 2\varepsilon' > 1 - \delta(\varepsilon/2) \end{aligned}$$

and so  $\|S'_k z_k - y\| < \varepsilon/2$ . Since  $H$  is a Hilbert space, for each  $k \in A$ , there exists an isometry  $R_k : H \rightarrow H$  such that  $R_k(S'_k z_k) = y$  and  $\|R_k - \operatorname{Id}\| < \varepsilon/2$  where  $\operatorname{Id}$  is the identity. We clearly have that  $A$ ,  $S_k = R_k \circ S'_k$  and  $z_k$  are the desired objects.  $\square$

**Remark 10.** We do not know whether every uniformly convex space can do the same role of that of Hilbert spaces in the above result.

As particular case of Proposition 9, we get the following example.

**Example 11.** Let  $K$  be a compact Hausdorff topological space and let  $H$  be a Hilbert space. Then the pair  $(\ell_1(C(K)), H)$  has the BPBp.

The proof reduces to use Proposition 9 together with the fact the  $(C(K), H)$  has the BPBp (see [16, Theorem 2.2] for the real case and [2, Theorem 2.4] for the complex case).

Another particular case is the following result, which follows from the fact that if  $X$  is a uniformly convex space, then the pair  $(X, Y)$  has the BPBp for every Banach space  $Y$  [15].

**Example 12.** Let  $X$  be a uniformly convex space and let  $H$  be a Hilbert space. Then the pair  $(\ell_1(X), H)$  has the BPBp. In particular,  $(\ell_1(L_p(\mu)), H)$  has the BPBp for every measure  $\mu$  and every  $1 < p < \infty$ . In fact, the same results hold when  $p = 1$  or  $p = \infty$ , since both  $(L_1(\mu), H)$  and  $(L_\infty(\mu), H)$  have the BPBp (see [11, 14] and [17], respectively.)

Our last result deals with pairs of Banach spaces whose second coordinate is a uniform algebra. We recall that a uniform algebra is a closed subalgebra of a  $C(K)$  space, equipped with the supremum norm, that separates the points of  $K$ . It is shown in [10, Theorem 3.6] that for every Asplund space  $X$  and every uniform algebra  $A$  of a  $C(K)$  space, the pair  $(X, A)$  has the BPBp with a function  $\varepsilon \mapsto \eta(\varepsilon)$  which does not depend neither on  $X$  nor on  $A$ . For every  $n \in \mathbb{N}$ , the space  $\ell_1^n(X)$  is also Asplund, so we get that  $(\ell_1^n(X), A)$  has the BPBp with the function  $\eta$ . Therefore, our Theorem 6 provides the following result.



**Corollary 13.** *Let  $X$  be an Asplund space and let  $A$  be a uniform algebra. Then the pair  $(\ell_1(X), A)$  has the BPBp.*

The spaces  $C_0(L)$  are uniform algebras for every locally compact space  $L$ . In this case, we may use [4, Theorem 2.4] instead of [10, Theorem 3.6].

**Example 14.** Let  $X$  be an Asplund space and let  $L$  be a locally compact Hausdorff topological space. Then the pair  $(\ell_1(X), C_0(L))$  has the BPBp.

We finish the paper with an open problem. It follows from Remark 5 and Theorem 6 that if  $(\ell_1(X), Y)$  has the BPBp, then both  $(X, Y)$  and  $(\ell_1, Y)$  have the BPBp. None of these two properties alone imply that  $(\ell_1(X), Y)$  has the BPBp. Indeed, if  $X$  is one-dimensional and  $Y$  fails the AHSP, then  $(X, Y)$  has the BPBp but  $(\ell_1(X), Y)$  fails it; on the other hand,  $X = L_1[0, 1]$  and  $Y = C[0, 1]$  satisfy that  $(\ell_1, Y)$  has the BPBp [3] but  $(\ell_1(X), Y) \equiv (X, Y)$  does not [1]. We do not know whether both properties together are enough to provide a reverse result.

**Question 15.** Let  $X, Y$  be Banach spaces. Suppose  $(X, Y)$  and  $(\ell_1, Y)$  have the BPBp. Does  $(\ell_1(X), Y)$  have the BPBp? Equivalently, if  $(X, Y)$  has the BPBp and  $Y$  has the AHSP, does  $(X, Y)$  have the generalized AHSP?

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